Useful Equations

- **Frequency/Angular Frequency**: \( \omega = 2\pi f = \frac{2\pi}{T} \)
- **Natural Frequency (Spring)**: \( \omega_0 = \sqrt{\frac{k}{m}} \)
- **Natural Frequency (Pendulum)**: \( \omega_0 = \sqrt{\frac{g}{l}} \)
- **Harmonic Oscillator Solution**: \( x(t) = A \cos(\omega_0 t + \phi) \)
- **Energy of Simple Harmonic Oscillator**: \( E = \frac{1}{2} k A^2 \)

Problems:

1. (*) Using the plot below of the position of a mass on a spring, determine the approximate maximum amplitude, maximum velocity, maximum acceleration, period, and phase angle of the motion.

   ![Plot of position vs. time](image)

   - We can read the amplitude from the plot, it oscillates between -5 and +5, so \( A = 5 \).
   - The period of this function appears to be about 6 (it is actually \( 2\pi \)), thus the angular velocity is \( \omega = \frac{2\pi}{T} \rightarrow \omega = 1 \) or 1.05.
   - We know that the maximum velocity and maximum acceleration are \( \left| v_{\text{max}} \right| = A \omega = 5 \text{ m/s} \) and \( \left| a_{\text{max}} \right| = A \omega^2 = 5 \text{ m/s}^2 \).
   - The phase angle is -\( \pi \) or +\( \pi \).

2. (***) Determine the equation of motion for each of the following scenarios. Each has a mass \( m = 10 \text{ kg} \) and spring constant \( k = 1,000 \text{ N/m} \). The equilibrium position is at \( x = 0 \).

   - (a) The mass is released from rest at a position -1 m.
     - \( A = 1 \text{ m} \)
     - \( \cos \phi = -1 \rightarrow \phi = \pi \)
     - \( x(t) = (1 \text{ m}) \cos((10 \text{ rad/s})t + \pi) \)
   
   - (b) The mass is pushed with a velocity 10 m/s at the origin in the positive x direction.
     - \( \cos \phi = 0 \ ( \phi = \pm \pi/2 ) \), \( A \omega = 10 \text{ m/s} \rightarrow \sin \phi = -1 \) and \( \phi = -\pi/2 \).
Thus \( A = \frac{10 \text{ m/s}}{10 \text{ rad/s}} = 1 \text{ m} \) \( \rightarrow \) \( x(t) = (1 \text{ m}) \cos \left( (10 \text{ rad/s}) t - \frac{\pi}{2} \right) \)

(c) The mass is thrown towards the origin from the position -1 m with an initial velocity of 10 m/s.

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\begin{align*}
-1 & = A \cos \phi \\
(10 \text{ m/s}) & = -A \omega \sin \phi = -A(10 \text{ rad/s}) \sin \phi \\
\cos \phi & = -\frac{1}{A} \quad \text{and} \quad \sin \phi = -\frac{1}{A} \quad \rightarrow \quad \text{cos} \phi = \sin \phi \quad \rightarrow \quad \phi = \frac{\pi}{4}
\end{align*}
\]

\[
A = -\frac{1}{\cos \phi} = -\sqrt{2}
\]

\[ x(t) = (1.414 \text{ m}) \cos \left( (10 \text{ rad/s}) t + \frac{\pi}{4} \right) \]

3. (**) Consider the Energy curve shown below. If this system represents a mass \( m = 5 \) kg oscillating on a spring of spring constant \( k = 2,000 \text{ Nm} \). A mass is released from rest at the position \( x = 2 \) m. Determine the equation describing the motion of the mass.

![Energy curve](image)

We can see that the equilibrium position of this mass appears to be about 5m. If the mass is released from the position \( x = 2 \) m at rest, then the turning points are located at 2 m and 8 m (an amplitude of 3 m).

If the mass is released with zero velocity at time \( t = 0 \), then \( v(t = 0) = A \omega \sin \phi = 0 \) which implies that \( \phi = 0, \pi \). Further, the spring begins compressed, \( x(t) = A \cos \phi = -A \) which implies that \( \phi = \pi \).

Further, calculation \( \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{(2,000 \text{ Nm})}{(5 \text{ kg})}} = 20 \text{ rad/s} \)

Thus the equation of motion is \( x(t) = (3 \text{ m}) \cos \left( (20 \text{ rad/s}) t + \pi \right) + 5 \text{ m} \)

4. (**) A mass \( m = 1 \text{ kg} \) is oscillating according to the formula \( x(t) = 4 \cos \left( 2\pi t - \frac{\pi}{4} \right) \). What is the total mechanical energy of this oscillator?

First, we can see clearly that \( A = 4 \text{ m} \) and \( \omega = 2\pi \).

From the frequency we can get \( k = \omega^2 m = (2\pi)^2 (1 \text{ kg}) = 4\pi^2 \text{ Nm} \).

The total energy of a harmonic oscillator is \( E = \frac{1}{2} k A^2 = \frac{1}{2} (4\pi^2)(4)^2 = 32\pi^2 \approx 316 \).