Topics:

- Homogeneous linear second-order constant coefficients equations
- Nonhomogeneous linear second-order constant coefficients equations
- Application of second-order constant coefficients equations to higher order linear constant coefficients equations
- Cauchy-Euler equations
- Spring-mass system modeling
MATH 267 SI Exam 2 Review

**Given:**

- Some solutions to
  \[ a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0 \]
  are of the form
  \[ e^{mx}, \]
  where
  \[ a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0 = 0 \]

- Certain quartic polynomials can be factored using the substitution \( z = x^2 \); e.g.,
  \[ x^4 - 1 = z^2 - 1 = (z - 1)(z + 1) = (x^2 - 1)(x^2 + 1) \]

- Given that
  \[ y = c_1 y_1 + c_2 y_2 \]
  is a general solution to
  \[ y'' + P(x)y' + Q(x)y = 0, \]
  then
  \[ y_p = u_1 y_1 + u_2 y_2 \]
  is a particular solution to
  \[ y'' + P(x)y' + Q(x)y = f(x), \]
  where
  \[ u_1' = -\frac{y_2 f(x)}{y_1 y_2' - y_1' y_2} \quad \text{and} \quad u_2' = \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2} \]

- Some solutions to
  \[ ax^2 y'' + bxy' + cy = 0 \]
  are of the form
  \[ x^m, \]
  where
  \[ am(m - 1) + bm + c = 0. \]

- Free damped motion can be modeled by
  \[ \frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0 \]
  The system is said to be
  - **Overdamped** if \( \lambda^2 - \omega^2 > 0 \)
  - **Critically damped** if \( \lambda^2 - \omega^2 = 0 \)
  - **Underdamped** if \( \lambda^2 - \omega^2 < 0 \)
Warm-up Problem

(a) If $(2r - 1)^2(4r^2 - 1)(4r^2 + 1) = 0$ is the auxiliary (characteristic) equation of the 6th order homogeneous differential equation then write down the general solution of the differential equation.

(b) If the 6th order differential equation in part (a) is non-homogeneous with $f(x) = xe^{x/2} + e^{-x/2} + x \sin(x/2)$ on the right hand side of the equation, write down a form of particular solution. You do not need to solve for constants.

PROBLEM 1

Use the reduction of order to find a second solution for the differential equation

$$y'' - \frac{5}{t} y' + \frac{8}{t^2} y = 0$$

Given that a solution is given by $y_1 = t^4$.

Write the general solution of this equation and the solution corresponding to initial conditions

$y(1) = 0, \quad y'(1) = 1$

PROBLEM 2

Find the solution of the boundary value problem

$$y''' - y'' + y' - y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1.$$  
(Hint: $m=1$ is a root of the associated auxiliary equation)

PROBLEM 3

Solve the differential equation $y''' + 4y' + 4y = 2016e^{-2x}$.

PROBLEM 4

Find the solution of the boundary value problem

$$y'' - y = e^x, \quad y(0) = 0, \quad y'(0) = 1$$
PROBLEM 5

Use the method of undetermined coefficients to find a particular solution of the differential equation

\[ y'' - y = xe^x + \cos(x) \]

PROBLEM 6

Solve \( y'' - 4y' - 12y = 3e^{5t} \) using both undetermined coefficients and variation of parameters methods.

PROBLEM 7

Use the method of variation of parameters to find a particular solution of the differential equation

\[ y'' + y = \frac{1}{\cos^3(x)} \quad x > 0. \]

Then write down the expression of the general solution of the equation, and the solution corresponding to initial condition \( y(0) = 0, y'(0) = 0 \).

PROBLEM 8

Determine the general solution to \( x^2 y'' + 3xy' + y = 1 \)

PROBLEM 9

Solve \( x^2 y'' - xy' + y = 2x \)

PROBLEM 10

A mass weighing 12 pounds stretches a spring 2 feet. The mass is initially released from a point 1 foot below the equilibrium position with an upward velocity of 4 ft/s. What are the amplitude, period of the simple harmonic motion? At what times do the mass return to the point 1 foot below the equilibrium position?