appearing in flowing fluids. Chapter 4 deals with the basic quantitative laws and equations of fluid flow. Chapter 5 treats flow of incompressible fluids through pipes and in thin layers, Chap. 6 is on compressible fluids in flow, and Chap. 7 describes flow past solids immersed in the flowing fluid. Chapter 8 deals with the important engineering tasks of moving fluids through process equipment and of measuring and controlling fluids in flow. Finally, Chap. 9 covers mixing, agitation, and dispersion operations, which in essence are applied fluid mechanics.

CHAPTER 2

FLUID STATICS AND ITS APPLICATIONS

NATURE OF FLUIDS. A fluid is a substance that does not permanently resist distortion. An attempt to change the shape of a mass of fluid results in layers of fluid sliding over one another until a new shape is attained. During the change in shape, shear stresses exist, the magnitudes of which depend upon the viscosity of the fluid and the rate of sliding, but when a final shape has been reached, all shear stresses will have disappeared. A fluid in equilibrium is free from shear stresses.

At a given temperature and pressure, a fluid possesses a definite density, which in engineering practice is usually measured in kilograms per cubic meter or pounds per cubic foot. Although the density of all fluids depends on the temperature and pressure, the variation in density with changes in these variables may be small or large. If the density changes only slightly with moderate changes in temperature and pressure, the fluid is said to be incompressible; if the changes in density are significant, the fluid is said to be compressible. Liquids are generally considered to be incompressible and gases compressible. The terms are relative, however, and the density of a liquid can change appreciably if pressure and temperature are changed over wide limits. Also, gases subjected to small percentage changes in pressure and temperature act as incompressible fluids, and density changes under such conditions may be neglected without serious error.
PRESSURE CONCEPT. The basic property of a static fluid is pressure. Pressure is familiar as a surface force exerted by a fluid against the walls of its container. Pressure also exists at every point within a volume of fluid. A fundamental question is: What kind of quantity is pressure? Is pressure independent of direction, or does it vary with direction? For a static fluid, as shown by the following analysis, pressure turns out to be independent of the orientation of any internal surface on which the pressure is assumed to act.

Choose any point O in a mass of static fluid and, as shown in Fig. 2.1, construct a cartesian system of coordinate axes with O as the origin. The x and y axes are in the horizontal plane, and the z axis points vertically upward. Construct a plane ABC cutting the x, y, and z axes at distances from the origin of Δx, Δy, and Δz, respectively. Planes ABC, AOC, COB, and AOB form a tetrahedron. Let θ be the angle between planes ABC and COB. This angle is less than 90° but otherwise is chosen at random. Imagine that the tetrahedron is isolated as a free body and consider all forces acting on it in the direction of the z axis, either from outside the fluid or from the surrounding fluid. Three forces are involved: (1) the force of gravity acting downward, (2) the pressure force on plane COB acting upward, and (3) the vertical component of the pressure force on plane ABC acting downward. Since the fluid is in equilibrium, the resultant of these forces is zero. Also, since a fluid in equilibrium cannot support shear stresses, all pressure forces are normal to the surface on which they act. Otherwise there would be shear-force components parallel to the faces.

The area of face COB is ΔxΔy/2. Let the average pressure on this face be \( \bar{p} \). Then the upward force on the face is \( \bar{p} Δx Δy/2 \). Let \( \bar{p} \) be the average pressure on face ABC. The area of this face is \( Δx Δy/2 \cos θ \), and the total force on the face is \( \bar{p} Δx Δy (2 \cos θ) \). The angle between the force vector of pressure \( \bar{p} \) with the z axis is also θ, so the vertical component of this force acting downward is

\[ \frac{\bar{p} Δx Δy \cos θ}{2 \cos θ} = \frac{\bar{p} Δx Δy}{2} \]

The volume of the tetrahedron is \( Δx Δy Δz/6 \). If the fluid density is \( ρ \), the force of gravity acting on the fluid in the tetrahedron is \( ρ Δx Δy Δz g/6g_c \). This component acts downward. The force balance in the z direction becomes

\[ \frac{\bar{p}}{2} Δx Δy - \frac{\bar{p}}{2} Δx Δy - \frac{ρ Δx Δy Δz g}{6g_c} = 0 \]

Dividing by \( Δx Δy \) gives

\[ \frac{\bar{p}}{2} - \frac{ρ Δz g}{6g_c} = 0 \]  \hspace{1cm} (2.1)

Now, keeping angle θ constant, let plane ABC move toward the origin O. As the distance between ABC and O approaches zero as a limit, ΔZ approaches zero also and the gravity term vanishes. Also, the average pressures \( \bar{p} \) and \( \bar{p}_x \) approach \( p \) and \( p_x \), the local pressures at point O, and Eq. (2.1) shows that \( p_x \) becomes equal to \( p \).

Force balances parallel to the x and y axes can also be written. Each gives an equation like Eq. (2.1) but containing no gravity term, and in the limit

\[ p_x = p \]

Since both point O and angle θ were chosen at random, the pressure at any point is independent of direction.

Fine powdery solids resemble fluids in many respects but differ considerably in others. For one thing, a static mass of particulate solids, as shown in Chap. 28, can support shear stresses of considerable magnitude and the pressure is not the same in all directions.

HYDROSTATIC EQUILIBRIUM. In a stationary mass of a single static fluid, the pressure is constant in any cross section parallel to the earth's surface but varies from height to height. Consider the vertical column of fluid shown in Fig. 2.2. Assume the cross-sectional area of the column is S. At a height Z above the base
or, between the two definite heights $Z_s$ and $Z_a$ shown in Fig. 2.2,
\[
\frac{P_s - P_a}{\rho} = \frac{\theta}{\rho g_c} (Z_s - Z_a)
\]  
(2.5)

Equation (2.4) expresses mathematically the condition of hydrostatic equilibrium.

**BAROMETRIC EQUATION.** For an ideal gas the density and pressure are related by the equation

\[
\rho = \frac{pM}{RT}
\]  
(2.6)

where $M$ = molecular weight

$T$ = absolute temperature

Substitution from Eq. (2.6) into Eq. (2.3) gives

\[
\frac{dp}{\rho} + \frac{gM}{g_c RT} dZ = 0
\]  
(2.7)

Integration of Eq. (2.7) between levels $a$ and $b$, on the assumption that $T$ is constant, gives

\[
\ln \frac{P_b}{P_a} = -\frac{gM}{g_c RT} (Z_b - Z_a)
\]  
(2.8)

Equation (2.8) is known as the *barometric equation*.

Methods are available in the literature for estimating the pressure distribution in situations, for example, a deep gas well, in which the gas is not ideal and the temperature is not constant.

As stated on page 14, these equations are used in SI units by setting $g_c = 1$.

**HYDROSTATIC EQUILIBRIUM IN A CENTRIFUGAL FIELD.** In a rotating centrifuge a layer of liquid is thrown outward from the axis of rotation and is held against the wall of the bowl by centrifugal force. The free surface of the liquid takes the shape of a paraboloid of revolution, but in industrial centrifuges the rotational speed is so high and the centrifugal force so much greater than the force of gravity that the liquid surface is virtually cylindrical and coaxial with the axis of rotation. This situation is illustrated in Fig. 2.3, in which $r_f$ is the radial distance from the axis of rotation to the free liquid surface and $r_c$ is the radius of the centrifuge bowl. The entire mass of liquid indicated in Fig. 2.3 is rotating like a rigid body, with no sliding of one layer of liquid over another. Under these conditions the pressure distribution in the liquid may be found from the principles of fluid statics.
The pressure drop over any ring of rotating liquid is calculated as follows. Consider the ring of liquid shown in Fig. 2.3 and the volume element of thickness $dr$ at a radius $r$:

$$dF = \frac{\omega^2 r \, dm}{g_c}$$

where $dF$ = centrifugal force
$dm$ = mass of liquid in element
$\omega$ = angular velocity, rad/s

If $\rho$ is the density of the liquid and $b$ the breadth of the ring,

$$dm = 2\pi r b dr$$

Eliminating $dm$ gives

$$dF = \frac{2\pi \rho b \omega^2 r^2}{g_c} \, dr$$

The change in pressure over the element is the force exerted by the element of liquid divided by the area of the ring:

$$dp = \frac{dF}{2\pi rb} = \frac{\omega^2 \rho r \, dr}{g_c}$$

The pressure drop over the entire ring is

$$p_2 - p_1 = \int_{Z_1}^{Z_2} \frac{\omega^2 \rho r \, dr}{g_c}$$

Assuming the density is constant and integrating gives

$$p_2 - p_1 = \frac{\omega^2 \rho (r_2^3 - r_1^3)}{2g_c}$$

(2.9)

MANOMETERS. The manometer is an important device for measuring pressure differences. Figure 2.4 shows the simplest form of manometer. Assume that the shaded portion of the U tube is filled with liquid $A$ having a density $\rho_A$ and that the arms of the U tube above the liquid are filled with fluid $B$ having a density $\rho_B$. Fluid $B$ is immiscible with liquid $A$ and less dense than $A$; it is often a gas such as air or nitrogen.

A pressure $p_2$ is exerted in one arm of the U tube and a pressure $p_1$ in the other. As a result of the difference in pressure $p_2 - p_1$, the meniscus in one branch of the U tube is higher than in the other, and the vertical distance between the two meniscuses $R_m$ may be used to measure the difference in pressure. To derive a relationship between $p_2 - p_1$ and $R_m$, start at the point 1, where the pressure is $p_1$; then, as shown by Eq. (2.4), the pressure at point 2 is $p_2 + (\gamma/g)(R_m + R_0)p_B$. By the principles of hydrostatics, this is also the pressure at point 3. The pressure at point 4 is less than that at point 3 by the amount $(\gamma/g)R_0p_A$, and the pressure at point 5, which is $p_5$, is still less by the amount $(\gamma/g)Z_0p_B$. These statements can be summarized by the equation:

$$p_2 + \frac{\gamma}{g_c}[(Z_m + R_0)p_B - R_0\rho_A - Z_0\rho_B] = p_5$$

Simplification of this equation gives

$$p_2 - p_1 = \frac{\gamma}{g_c} R_0(\rho_A - \rho_B)$$

(2.10)

Note that this relationship is independent of the distance $Z_m$ and of the dimensions of the tube provided that pressures $p_1$ and $p_2$ are measured in the same horizontal plane. If fluid $B$ is a gas, $\rho_B$ is usually negligible compared to $\rho_A$ and may be omitted from Eq. (2.10).
Example 21. A manometer of the type shown in Fig. 2.4 is used to measure the pressure drop across an orifice (see Fig. 8.19). Liquid A is mercury (density 13,590 kg/m³) and fluid B, flowing through the orifice and filling the manometer leads, is brine (density 1260 kg/m³). When the pressures at the taps are equal, the level of the mercury in the manometer is 0.9 m below the orifice taps. Under operating conditions, the gauge pressure† at the upstream tap is 0.14 bar; the pressure at the downstream tap is 250 mm Hg below atmospheric. What is the reading of the manometer in millimeters?

Solution

Call atmospheric pressure zero and note that $g_e = 1$; then the numerical data for substitution in Eq. (2.10) are

$$p_a = 0.14 \times 10^5 = 14000 \text{ Pa}$$

From Eq. (2.5)

$$p_b = Z_A g_A (g_e)$$

$$= -250/1000 \times (9.80665/1) \times 13590$$

$$= -33318 \text{ Pa}$$

Substituting in Eq. (2.10) gives

$$14000 + 33318 = R_a \times 9.80665 \times (13590 - 1260)$$

$$R_a = 0.391 \text{ m or 391 mm}$$

For measuring small differences in pressure, the inclined manometer shown in Fig. 2.5 may be used. In this type, one leg of the manometer is inclined in such a manner that, for a small magnitude of $R_a$, the meniscus in the inclined tube must move a considerable distance along the tube. This distance is $R_a$ divided by the sine of $\alpha$, the angle of inclination. By making $\alpha$ small, the magnitude of $R_a$ is multiplied into a long distance $R_1$, and a large reading becomes equivalent to a small pressure difference; so

$$p_a - p_b = \frac{g}{g_e} R_1 (\rho_A - \rho_B) \sin \alpha$$

(2.11)

In this type of pressure gauge, it is necessary to provide an enlargement in the vertical leg so that the movement of the meniscus in the enlargement is negligible within the operating range of the instrument.

CONTINUOUS GRAVITY DECANTER. A gravity decanter of the type shown in Fig. 2.6 is used for the continuous separation of two immiscible liquids of differing densities. The feed mixture enters at one end of the separator; the two liquids flow slowly through the vessel, separate into two layers, and discharge through overflow lines at the other end of the separator.

Provided the overflow lines are so large that frictional resistance to the flow of the liquids is negligible, and provided they discharge at the same pressure as that in the gas space above the liquid in the vessel, the performance of the decanter can be analyzed by the principles of fluid statics.

For example, in the decanter shown in Fig. 2.6 let the density of the heavy liquid be $\rho_A$ and that of the light liquid be $\rho_B$. The depth of the layer of heavy liquid is $Z_{AS}$ and that of the light liquid is $Z_B$. The total depth of liquid in the vessel $Z_T$ is fixed by the position of the overflow line for the light liquid. Heavy liquid discharges through an overflow leg connected to the bottom of the vessel and rising to a height $Z_{AS}$ above the vessel floor. The overflow lines and the top of the vessel are all vented to the atmosphere.

† Gauge pressure is pressure measured above the prevailing atmospheric pressure.
Since there is negligible frictional resistance to flow in the discharge lines, the column of heavy liquid in the heavy-liquid overflow leg must balance the somewhat greater depth of the two liquids in the vessel. A hydrostatic balance leads to the equation

\[ Z_{A0} \rho_B + Z_{A1} \rho_A = Z_{A2} \rho_B \]  

(2.12)

Solving Eq. (2.12) for \(Z_{A1}\) gives

\[ Z_{A1} = Z_{A2} - Z_B \frac{\rho_B}{\rho_A} = Z_{A2} - (Z_T - Z_{A1}) \frac{\rho_B}{\rho_A} \]  

(2.13)

where the total depth of liquid in the vessel is \(Z_T = Z_B + Z_{A1}\). From this

\[ Z_{A1} = \frac{Z_{A2} - Z_T \frac{\rho_B}{\rho_A}}{1 - \frac{\rho_B}{\rho_A}} \]  

(2.14)

Equation (2.14) shows that the position of the liquid-liquid interface in the separator depends on the ratio of the densities of the two liquids and on the elevations of the overflow lines. It is independent of the rates of flow of the liquids. Equation (2.14) shows that as \(\rho_A\) approaches \(\rho_B\), the position of the interface becomes very sensitive to changes in \(Z_{A2}\), the height of the heavy-liquid leg. With liquids that differ widely in density this height is not critical, but with liquids of nearly the same density it must be set with care. Often the top of the leg is made movable so that in service it can be adjusted to give the best separation.

The size of a decanter is established by the time required for separation, which in turn depends on the difference between the densities of the two liquids and on the viscosity of the continuous phase. Provided the liquids are clean and do not form emulsions, the separation time may be estimated from the empirical equation

\[ t = \frac{100 \mu}{\rho_A - \rho_B} \]  

(2.15)

where \(t\) = separation time, h
\(\rho_A, \rho_B\) = densities of liquids \(A\) and \(B\), kg/m³
\(\mu\) = viscosity of the continuous phase, cP

Equation (2.15) is not dimensionless, and the indicated units must be used.

**Example 2.2.** A horizontal cylindrical continuous decanter is to separate 1500 bbl/d (day) (593 m³/h) of a liquid petroleum fraction from an equal volume of wash water. The oil is the continuous phase and at the operating temperature has a viscosity of 1.1 cP and a density of 54 lb/ft³ (865 kg/m³). The density of the acid is 72 lb/ft³ (1153 kg/m³). Compute (a) the size of the vessel, and (b) the height of the acid overflow above the vessel floor.

**Solution**

(a) The vessel size is found from the separation time. Substitution in Eq. (2.15) gives

\[ t = \frac{100 \times 1.1}{1153 - 865} = 0.38 \text{ h} \]
In operation of the machine the feed is admitted continuously near the bottom of the bowl. Light-liquid discharges at point 2 through ports near the axis of the bowl; heavy liquid passes under a ring, in the direction toward the axis of rotation, and discharges over a dam at point 1. If there is negligible frictional resistance to the flow of the liquids as they leave the bowl, the position of the liquid-liquid interface is established by a hydrostatic balance and the relative “heights” (radial distances from the axis) of the overflow ports at 1 and 2.

Assume that the heavy liquid, of density \( \rho_A \), overflows the dam at radius \( r_A \), and the light liquid, of density \( \rho_B \), leaves through ports at radius \( r_h \). Then if both liquids rotate with the bowl and friction is negligible, the pressure difference in the light liquid between \( r_B \) and \( r_h \) must equal that in the heavy liquid between \( r_A \) and \( r_h \). The principle is exactly the same as in a continuous gravity decanter.

Thus

\[
p_l - p_B = p_A - p_h
\]

(2.16)

where

\[ p_l = \text{pressure at liquid-liquid interface} \]
\[ p_h = \text{pressure at free surface of light liquid at } r_h \]
\[ p_A = \text{pressure at free surface of heavy liquid at } r_A \]

From Eq. (2.9)

\[ p_l - p_h = \frac{\alpha^2 p_h (r_h^2 - r_B^2)}{2 \eta} \quad \text{and} \quad p_h - p_A = \frac{\omega^2 \rho_A (r_h^2 - r_A^2)}{2 \eta} \]

Equating these pressure drops and simplifying leads to

\[ \rho_B (r_h^2 - r_B^2) = \rho_A (r_h^2 - r_A^2) \]

Solving for \( r_h \), gives

\[
r_h = \sqrt{r_A^2 - \left( \frac{\rho_B}{\rho_A} \right)^2 r_h^2} - \sqrt{r_A^2 - \left( \frac{\rho_B}{\rho_A} \right)^2 r_h^2}
\]

(2.17)

Equation (2.17) is analogous to Eq. (2.14) for a gravity settling tank. It shows that \( r_h \), the radius of the neutral zone, is sensitive to the density ratio, especially when the ratio is nearly unity. If the densities of the fluids are too nearly alike, the neutral zone may be unstable even if the speed of rotation is sufficient to separate the liquids quickly. The difference between \( \rho_A \) and \( \rho_B \) should not be less than approximately 3 percent for stable operation.

Equation (2.17) also shows that if \( r_B \) is held constant and \( r_A \), the radius of the discharge lip for the heavier liquid, increased, the neutral zone is shifted toward the wall of the bowl. If \( r_A \) is decreased, the zone is shifted toward the axis. An increase in \( r_B \), at constant \( r_A \), also shifts the neutral zone toward the axis, and a decrease in \( r_B \) causes a shift toward the wall. The position of the neutral zone is important practically. In zone \( A \), the lighter liquid is being removed from a mass of heavier liquid, and in zone \( B \), heavy liquid is being stripped from a mass of light liquid. If one of the processes is more difficult than the other, more time should be provided for the more difficult step. For example, if the separation in zone \( B \) is more difficult than that in zone \( A \), zone \( B \) should be large and zone \( A \) small. This is accomplished by moving the neutral zone toward the wall by increasing \( r_h \) or decreasing \( r_A \). To obtain a larger time factor in zone \( A \), the opposite adjustments would be made. Many centrifugal separators are so constructed that either \( r_h \) or \( r_B \) can be varied to control the position of the neutral zone.

FLOW THROUGH CONTINUOUS DECANTERS. Equations (2.14) and (2.17) for the interfacial position in continuous decanters are based entirely on hydrostatic balances. As long as there is negligible resistance to flow in the outlet pipes, the position of the interface is the same regardless of the rates of flow of the liquids and of the relative quantities of the two liquids in the feed. The rate of separation is the most important variable, for as mentioned before, it fixes the size of a gravity decanter and determines whether or not a high centrifugal force is needed. The rates of motion of a dispersed phase through a continuous phase are discussed in Chap. 7.

**SYMBOLS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>Breadth, m or ft</td>
</tr>
<tr>
<td>( F )</td>
<td>Force, N or lb</td>
</tr>
<tr>
<td>( g )</td>
<td>Gravitational acceleration, m/s(^2) or ft/s(^2)</td>
</tr>
<tr>
<td>( g_c )</td>
<td>Newton’s-law proportionality factor, 32.174 ft-lb/lbf-s(^2)</td>
</tr>
<tr>
<td>( M )</td>
<td>Molecular weight</td>
</tr>
<tr>
<td>( m )</td>
<td>Mass, kg or lb</td>
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2.6. A centrifuge bowl 250-mm ID (internal diameter) is turning at 4000 rpm. It contains a layer of chlorobenzene 50 mm thick. If the density of the chlorobenzene is 1109 kg/m³ and the pressure at the liquid surface is atmospheric, what gauge pressure is exerted on the wall of the centrifuge bowl?

2.7. The liquids described in Prob. 2.4 are to be separated in a tubular centrifuge bowl with an inside diameter of 150 mm, rotating at 8000 rpm. The free liquid surface inside the bowl is 40 mm from the axis of rotation. If the centrifuge bowl is to contain equal volumes of the two liquids, what should be the radial distance from the rotational axis to the top of the overflow dam for heavy liquid?

REFERENCES